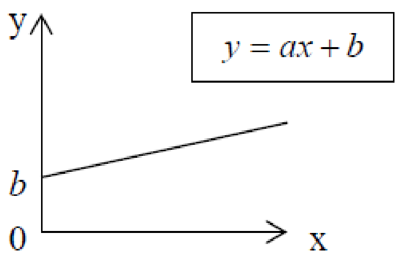
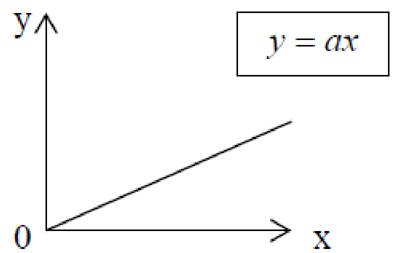
Linear Graphs and Linearization of Curved Graphs

1. Straight-line graphs

If the trend of the data is a straight line, then the graph is called linear.



Here, there is a *linear* relationship between y and x



This one is a special type of linear relationship between y and x, because b=0 so the line passes through the origin (0,0). Variable y is *directly proportional* to X. (Here, doubling x will double y.)

Data from experiments will never lie exactly on a line, but it is usually clear to the eye if the trend is linear or curved. If there is a linear relationship, the next step is usually to find (i) the slope *a* and (ii) the y intercept *b*. These two quantities fully describe the line.

Guidelines for plotting the points

a. Each graph must have your name, and the date
b. The graph must have a title
c. “widgets versus gizmos” means widgets on the vertical axis and gizmos on the horizontal axis
d. The axes must each be labeled with the name of the variable (e.g. speed), the symbol for the variable (e.g. v), and the units of the variable (e.g. m/s)
e. The scale of each axis must be chosen so that it’s easy to plot the points and to read off the coordinates of points. Use only multiples of 2 or 5 to label the bold grid lines on the graph paper. (e.g. 8, 10, 12, 14,…, but not 8, 10.5, 13, 15.5,…)
f. The axes don’t have to start at zero; the range should start near the value of the first point
g. Use one side of graph paper for each graph. The range of values for each axis must be selected so the data points use at least half of the extent of the axis.
h. The points must be tiny. Draw a circle around them to clarify where they are.

The best line:

To obtain the slope and y intercept, the *best line* must be drawn through the data. Although the slope can be calculated from the data points (using the method of least-squares), it can be done quite well by eye. Use a sufficiently long ruler and construct a *single* line that approximates the trend of the data points as accurately as possible. A transparent ruler is best. Don't pick up the pencil. Note that the line need not touch any of the actual data points. Sometimes, a point lies so far from the trend of the other points that it must be incorrect. This is called an *outlier* and should be ignored in finding the best line. Think: since you use all the acceptable points to interpolate the best line, it represents information "crystallized" from *all* the data.

The slope:

a. Select two new "slope points" on the best line. Indicate them by putting a square around each and number them 1 and 2. They must not be data points.
b. The slope points must be near the ends of the best line – i.e. far apart. Don't select them by looking for a point where the best line intersects with the grid lines
c. On the graph, label the slope points with their coordinates and units
d. The slope formula is:

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

e. Show the slope calculation (on the graph if possible) by first writing the above equation, then substituting the slope-point coordinates with units, and only then giving the answer.
f. The slope will be wrong if you leave out the unit. This is a common mistake!

The y intercept:

After you have the calculated numerical value of the slope *a* (with its unit), you can calculate the y intercept *b* of the point by solving the straight-line equation for *b*:

$$\boxed{b = y - ax}$$

To evaluate the right-hand side, use the coordinates (x,y) of any point on the best line, and not a data point.

Why do we never use the data points after the best line is found? The idea is that all the information comes from the best line, which contains more information than any one data point.

2 Example: Linear plot

We will plot a graph of gravitational potential energy versus height. Then, we will find the slope and the intercept.

Height h (meters)	Gravitational Potential Energy U (Joules)
0.20	0.49
0.40	1.04
0.60	1.33
0.80	1.96
1.00	2.48
1.20	3.21
1.40	3.43
1.60	3.93
1.80	4.54
2.00	1.73

As can be seen on the next page, once the data is plotted and a "best straight" line is found, the slope is determined to be 2.63 $^{J}/_{m}$ and the y-intercept is found to be $-0.10\ J$.

The theoretical expression of Potential Energy as a function of height can be written as

$$U_G = mgh$$

The equation of a line is

$$y = ax + b$$

Comparing the two equations we can make the associations:

$$U_G \rightarrow y \qquad h \rightarrow x \qquad mg \rightarrow a \qquad 0 \rightarrow b$$

Comparing the y-intercept, this is close $-0.10\ J$ is close to Zero. We can think of this as a measure of error in the experiment.

Comparing the slope term $mg = a$, if we assume we know g, the acceleration of gravity is the usually accepted value of 9.80 $^{m}/_{s^2}$, we can solve for the mass used in the experiment and we find that is

$$mg = a$$

$$m = \frac{a}{g} = \frac{2.63\ ^{J}/_{m}}{9.80\ ^{m}/_{s^2}} = 0.268\ \frac{J\ s^2}{m^2}\ x\ \frac{1\ kg\ ^{m^2}/_{s^2}}{J} = 0.268\ kg$$

So, the mass used in this experiment was 0.268 kg !

Gravitational Potential Energy
versus
height.

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4 Feb, 2008

Slope: $a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{U_2 - U_1}{h_2 - h_1}$

$= \frac{4.90\,J - 0.70\,J}{1.90\,m - 0.30\,m}$

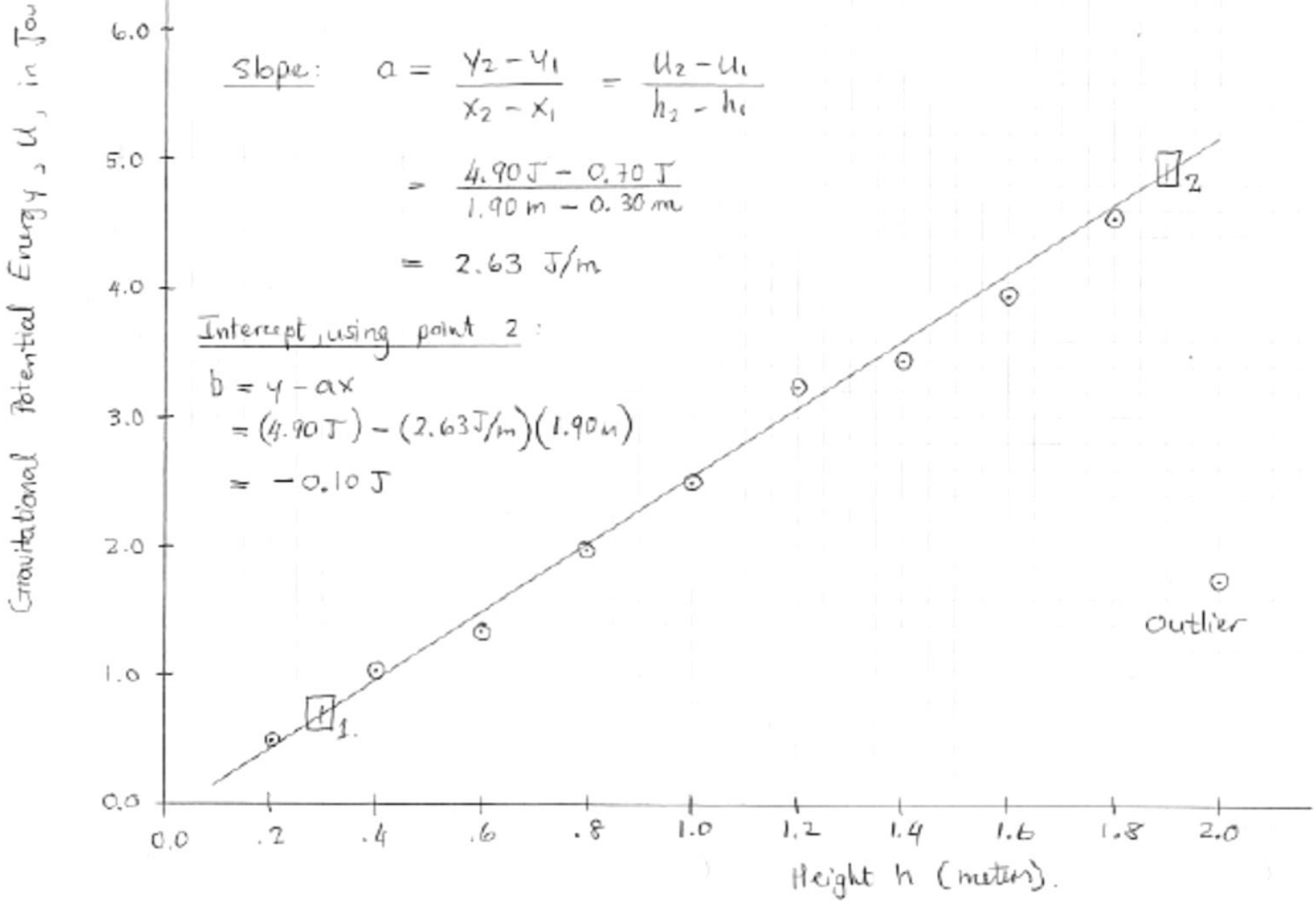
$= 2.63\ J/m$

Intercept, using point 2:

$b = y - ax$

$= (4.90\,J) - (2.63\,J/m)(1.90\,m)$

$= -0.10\,J$



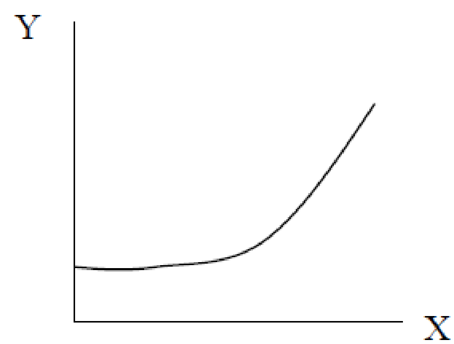
3. Exercise: A Linear Plot

The data tabulated here is hypothesized to have a linear trend. Make a graph of R versus v to verify this. Find the slope and the intercept. Follow the same methods as for the example above. Be sure to include units and proper formats for axis labels and plot titles which should include your name and the date you did the work.

v ($^{m}/_{s}$)	R ($^{kg}/_{s}$)
2.5	32.75
3.5	28.25
4.5	23.75
5.5	19.25
6.5	14.75
7.5	10.25
8.5	5.75
9.5	1.25
10.5	-3.25
11.5	-7.75

4. Curved Graphs

Often the plotted data will be curved. The eye can't tell one type of curve from another just by looking.



For example, it could be one of …

$Y = k\theta^n$ (Power-Law Relationship)

$Y = ke^{cX}$ (Exponential Relationship)

In the first equation, the plotted variables are Y and θ, and the other two quantities, k and n, are constants. In the second equation, the plotted variables are Y and X, and the constants are k and c. Also, recall that e is the base of the natural logarithm and has value 2.71828…

When experimental data are plotted, the numerical values of the constants are often needed to find the results. There are many different approaches to extracting the constants from a graph. We do it in a couple of ways by plotting a new graph that converts the data into a straight line. The constants can then be found by calculating the slope and intercept of this line.

4. A Parabola

A parabola is a power-law relationship where n=2, so the one variable is proportional to the square of the other.

The data in the table below is hypothesized to follow the equation $m = \frac{1}{2}qt^2$. Since measurements are always limited in their accuracy, you should not expect the plot to form a perfect parabola because of this randomness.

Here are two ways to find the value of the constant q. Both involve matching the plotted quantities with the x and y of the straight line,

$$y = ax + b$$

Time (sec)	Mass (kg)
2.0	4
3.0	21
4.0	33
5.0	43
6.0	74
7.0	90
8.0	126
9.0	163
10.0	194
11.0	230

4.1 Plotting *m* versus *t*²

Your instructor will discuss why we expect a graph of *m* versus t^2 to follow the trend of a straight line through the origin. If you plot the graph, the slope should be the value of the constant multiplying t^2, in other words: $slope = \frac{1}{2}q$. From this you can find the value of the constant q in the experiment.

4.2 Plotting $\sqrt{m}$ versus *t*

Your instructor will discuss why this is also expected to be a straight line if the hypothesis is correct. The constant q can be found again, but pay attention to the logic needed, since it is slightly different this time. The slope this time should be equal to $\sqrt{\frac{q}{2}}$.

5. Using Logarithms to Linearize Non-Linear Data

An alternative way to analyze power law relationship is the use of logarithms. This method is used heavily in determining what the best 'power' is for the power law relationship when a trustworthy theory is not available or you are testing this aspect of the theory. The method requires the taking of the natural log of both the independent and the dependent columns of data.

Let's consider the basic Power-Law equation we introduced earlier

$$Y = k\theta^n$$

Now take the natural log of both sides of this equation. We will get:

$$ln(Y) = ln(k\theta^n)$$

One of the rules of logarithms is the following identity holds

$$ln(AB) = ln(A) + ln(B)$$

So, our equation above becomes:

$$ln(Y) = ln(k) + ln(\theta^n)$$

Or we could write it

$$ln(Y) = ln(\theta^n) + ln(k)$$

A second rule of logarithms is this identity:

$$ln(A^B) = B\ ln(A)$$

This results in our equation being

$$ln(Y) = n\ ln(\theta) + ln(k)$$

Now if we compare this equation to the standard equation of a straight line we get

$$ln(Y) = n\ ln(\theta) + ln(k)$$
$$y = ax + b$$

If we make the following assignments we can see the two equations are similar

$$ln(Y) \to y$$
$$n \to a$$
$$ln(\theta) \to x$$
$$ln(k) \to b$$

So, if we plot $ln(Y)$ on the y axis and $ln(\theta)$ on the x axis, then the slope will be found to be *n* the power of the Power-Law. The y-intercept will be the $ln(k)$.

Consider our previous example that had the relationship: $m = \frac{1}{2}qt^2$

If we plot $ln(m)$on the y axis, $ln(t)$ on the x axis. Our slope should be close to 2. The y-intercept would be $b = ln\left(\frac{1}{2}q\right)$

We find q by using the equation

$$q = 2e^b$$

We get this from the fact that

$$b = ln\left(\frac{1}{2}q\right)$$

Take exponential of both sides

$$e^b = \frac{1}{2}q$$

Solve for q and you get

$$q = 2e^b$$

6. Exercises for Non-Linear Plots

Using our assumed relationship is

$$m = \frac{1}{2}qt^2$$

The data is

Time (sec)	Mass (kg)		
2.0	4		
3.0	21		
4.0	33		
5.0	43		
6.0	74		
7.0	90		
8.0	126		
9.0	163		
10.0	194		
11.0	230		

6.1 Linearization by Powers

Choose to linearize this by either plotting *m* versus t^2 or plotting $\sqrt{m}$ versus *t*

Fill in the two columns so that you can more easily create a plot that will be linear. Create the plot you choose and determine the slope and y-intercept of the "best straight line". Remember to use proper axis labels, and plot titles including your name and the date you did the work.

For whichever method you choose, write out the steps to allow you to solve for *q* from the determined slope and/or y-intercept. Put in the values and determine the value of q include units.

6.2 Use Logarithms to Linearize the data

Fill in the two columns to allow you to more easily create the ln-ln plot.

Time (sec)	Mass (kg)		
2.0	4		
3.0	21		
4.0	33		
5.0	43		
6.0	74		
7.0	90		
8.0	126		
9.0	163		
10.0	194		
11.0	230		

Create the plot and determine the slope and y-intercept of the "best straight line". Remember to use proper axis labels, and plot titles including your name and the date you did the work.

Write out the steps to allow you to solve for *q* from the determined slope and/or y-intercept. Put in the values and determine the value of q include units.

6.3 Compare Results

Finally, compare your values of q from the two different methods (Linearize by Powers or Linearize using Logarithms). Calculate a percentage difference using the formula

$$\%diff = \left|\frac{q_{Powers} - q_{Ln}}{q_{ave}}\right| \; x\; 100\% = 2\left|\frac{q_{Powers} - q_{Ln}}{q_{Powers} + q_{Ln}}\right| \; x\; 100\%$$